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Perturbation Theory for Hamilton's Principal Function

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A perturbation theory for solving two-point boundary value problems is presented. Based on Hamilton's principle and the calculus of variations, this theory analytically solves for the velocities in the targeting problem. Applications of the theory are found primarily in the fields of orbital mechanics and optimal control. Examples showing the accuracy of the theory in the two-body problem are presented.

I. Introduction

SOLUTIONS to two-point boundary value problems (2BVPs) are of significant importance in the field of astrodynamics and have been subject to extensive research over the years. However, the lack of procedures that automatically converge to the desired solution remains as a fundamental difficulty in solving these problems. Usually, solutions involve open-ended iterative methods that often have no guaranteed convergence and require a good initial guess. Examples of these methods include the method of homotopy, multiple shooting combined with Newton's iteration, and a variety of other techniques [1].

Even for many problems with known solutions, implicit equations must be solved iteratively to satisfy the 2BVP. In the two-body problem, this situation occurs in finding a solution to Lambert's problem, which involves iteration [2,3].

Guibout and Scheeres outlined a novel approach for solving 2BVPs using generating functions for canonical transformations [4]. Their approach requires the system to be a Hamiltonian dynamical system and relies on solving the Hamilton–Jacobi equation, which has its base in Hamilton's principle.

When a perturbation is present, the solution to the nominal problem is no longer valid (although it can be used as an initial guess to an iterative method), and one must resort to numerical methods. Conventional solutions to 2BVPs involve numerical integration schemes and open-ended iterative solutions. These types of solutions focus on finding an optimal transfer cost around the vicinity of the nominal solution. Moreover, with such an approach, one could easily find the needed correction to the initial impulse in order to hit the desired target. However, this technique will not portray an accurate description of the system and its behavior. It will concentrate on one target point and find the best solution. While in practice this is desirable, it does not obtain a detailed analysis of the vicinity of the desired solution.

Guibout and Scheeres [4] also suggested using Hamilton's principal function (HPF) to solve the 2BVP for the two-body problem. This function is derived directly from Hamilton's principle and yields solutions to the equations of motion through "simple differentiations and eliminations" [5]. Although the generating function of the canonical transformation and HPF have different physical significance, they are intimately related. HPF allows the initial and final endpoints and times to be dynamic; that is, they are free to change without affecting the structure of the function. Generating functions, on the other hand, has static initial conditions, viewed as constants of motion, with only the final endpoints being dynamic.

In this paper, an analytical perturbation technique is developed, and it solves the 2BVP of a perturbed system using HPF. This technique finds its applications in the two-body problem, configuration of spacecraft formations, optimal control problems, and a variety of problems in astrodynamics and other areas as well. This theory is applied to the problem of two bodies, and it can be used to obtain the solution to the system under perturbing forces.

For the nominal two-body 2BVP, Lambert's problem can be solved iteratively in order to obtain desired solutions; therefore, the solution to these types of problems is well established and generally straightforward. Using analytical expressions developed by the perturbation theory developed in this paper allows one to obtain a closed-form solution for the perturbed Lambert's problem. Therefore, a family of solutions in the vicinity of the desired target can be obtained. This fact helps us better understand the dynamics of the system, even when perturbing forces are present. Solutions to the perturbed system are obtained by analytically expanding HPF around the nominal solution using a small parameter.

Peñagaricano and Scheeres first published the results of the perturbation theory for the 2BVP, along with a perturbation theory for the initial value problem (IVP) [6]. However, the development of the theory was not carried out in detail, and only the results were published, whereas an in-depth derivation and description of the method is presented here. Additionally, examples that better illustrate the application of the perturbation theory are presented in this paper, along with applications to more complex problems. Numerical error analysis is also included in this paper along with an example, both of which were lacking in the previous publication.

The presented methodology is distinct from other analytical applications of HPF or generating functions. Two brief examples are presented in the following. For a given perturbed Hamiltonian system, von Zeipel's method can be used to reduce the Hamiltonian function through canonical transformations into a form independent of periodic perturbed coordinates [7]. The reduced Hamiltonian function depends only on the constants of motion of the perturbed system. The perturbation theory arises from the same basis as von Zeipel's method: through expansions around a small parameter. However, the presented method is used to solve the perturbed 2BVP, and no reduction in the Hamiltonian is sought.

The principal function also has a fundamental relation to action-angle variables for integrable systems. For example, in the two-body problem, HPF can be directly expressed in terms of these action-angle variables, as represented by the Delaunay elements. However, the expression of the principal function in terms of action variables is not suited to the solution of physically relevant 2BVPs that can arise in practical astrodynamics. Rather, this permits the theory presented

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in this paper to focus on solving the 2BVP by solving for the initial and final velocities that connect the initial and final position coordinates for a given flight time. The action-angle variables are better suited for solving the IVP.

II. Hamilton's Principal Function

In the first half of the 19th century, Sir William Rowan Hamilton proposed the existence of a fundamental function that yields the solution of conservative dynamical systems by simple differentiations and eliminations. This function is known as HPF, and Hamilton proved its existence in geometrical optics, and later in his study of dynamics [5]. To construct HPF, consider a system with the following Hamiltonian function:

$$H(\mathbf{q}, \mathbf{p}, t) = \dot{\mathbf{q}} \cdot \mathbf{p} - L(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{1}$$

where $\mathbf{q}$ and $\mathbf{p}$ are the generalized coordinates and generalized momenta, respectively, each of dimension n, and L is the Lagrangian of the system. The equations of motion of a Hamiltonian dynamical system can be expressed as functions of the Hamiltonian,

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \tag{2}$$

Consider now the action integral of a conservative dynamical system:

$$A = \int_{t_0}^{t_1} L(\mathbf{q}, \dot{\mathbf{q}})\, \mathrm{d}t \tag{3}$$

A solution for a Hamiltonian dynamical system exists if the action integral is stationary: it satisfies $\delta A = 0$. It can be shown that the condition for $\delta A = 0$ is equivalent to the path in phase space satisfying Eq. (2). Note that performing this variation also requires $\delta \mathbf{q} = 0$ and $\delta t = 0$, which means that

$$\delta \mathbf{q}(t_0) = \delta \mathbf{q}(t_1) = 0 \tag{4}$$

The action integral evaluated along a trajectory in phase space that yields a stationary value of A ($\delta A = 0$) is, by definition, HPF, and it is evaluated between times t_0 and t_1. Using Eq. (1), the action integral can be rewritten as

$$W(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1) = \int_{t_0}^{t_1} [\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p})]\, \mathrm{d}t \tag{5}$$

where it is assumed that $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})$. By definition, the action integral is equal to HPF, $W(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1)$, where $\mathbf{q}_0$ and $\mathbf{q}_1$ are the generalized coordinates at times t_0 and t_1, respectively. These two sets of generalized coordinates are connected by a path in phase space that makes the action integral stationary. Note that the principal function is defined by the endpoints and times. To derive the governing equations for W, consider a variation of the action integral's endpoints and times that yields

$$\delta W = \delta A = [\mathbf{p} \cdot \delta \mathbf{q} - H(\mathbf{q}, \mathbf{p}) \delta t]_{t_0}^{t_1} \tag{6}$$

On the other hand, if the variation of $W(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1)$ is taken, it leads to

$$\delta W = \frac{\partial W}{\partial \mathbf{q}_0} \cdot \delta \mathbf{q}_0 + \frac{\partial W}{\partial t_0} \delta t_0 + \frac{\partial W}{\partial \mathbf{q}_1} \cdot \delta \mathbf{q}_1 + \frac{\partial W}{\partial t_1} \delta t_1 \tag{7}$$

Equating Eqs. (6) and (7) yields

$$\mathbf{p}_0 = -\frac{\partial W}{\partial \mathbf{q}_0} \tag{8}$$

$$\mathbf{p}_1 = \frac{\partial W}{\partial \mathbf{q}_1} \tag{9}$$

and

$$-\frac{\partial W}{\partial t_0} + H(\mathbf{q}_0, \mathbf{p}_0, t_0) = 0 \qquad \frac{\partial W}{\partial t_1} + H(\mathbf{q}_1, \mathbf{p}_1, t_1) = 0 \tag{10}$$

The expressions in Eqs. (8) and (9) are the two boundary conditions that the principal function must satisfy, while Eq. (10) shows the pair of partial differential equations that must hold true for W. If the momenta in the Hamiltonian are substituted for, Eq. (10) becomes

$$-\frac{\partial W}{\partial t_0} + H\left(\mathbf{q}_0, -\frac{\partial W}{\partial \mathbf{q}_0}, t_0\right) = 0 \qquad \frac{\partial W}{\partial t_1} + H\left(\mathbf{q}_1, \frac{\partial W}{\partial \mathbf{q}_1}, t_1\right) = 0 \tag{11}$$

The preceding partial differential equations define HPF and are discussed in more detail in the thesis by Guibout [8]. With the knowledge of the $W(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1)$ function, Eq. (8) can be used to solve for $\mathbf{q}_1 = \mathbf{q}_1(\mathbf{q}_0, \mathbf{p}_0)$. This solution leads in turn to a solution of Eq. (9), and hence to expressions for both $\mathbf{q}_1$, and $\mathbf{p}_1$, which permit solution of the IVP. Therefore, Hamilton's principle encompasses the dynamics of the system by transforming a set of initial conditions $(\mathbf{q}_0, \mathbf{p}_0, t_0)$ into $(\mathbf{q}_1, \mathbf{p}_1, t_1)$.

However, in this paper, HPF is used to solve the 2BVP. With the knowledge of the W function and the initial and final endpoints and times $(\mathbf{q}_0, t_0)$, $(\mathbf{q}_1, t_1)$, the initial and final momenta that solve the problem can be obtained through differentiation in Eqs. (8) and (9). In the following section, it is shown how this method can be used to solve the perturbed 2BVP using the nominal dynamics of the system.

III. Perturbation Theory for Hamilton's Principal Function

Hamilton developed a first-order perturbation theory for a general Hamiltonian dynamical system [9]. However, Hamilton's perturbation method is developed for the first order and makes assumptions about the system, and it is not stated in general terms. The reader is referred to Appendix A for a derivation of Hamilton's perturbation theory.

In the following, a complete perturbation method is derived starting from Hamilton's outline. Consider the generalized coordinate and momentum vectors of a $2n$-dimensional Hamiltonian dynamical system:

$$\mathbf{q} = \sum_{i=1}^{n} q_i \delta \mathbf{q} \qquad \mathbf{p} = \sum_{i=1}^{n} p_i \delta \mathbf{p} \tag{12}$$

where q_i and p_i are the ith direction components of $\mathbf{q}$ and $\mathbf{p}$, respectively. Following the Einstein convention, δq_i and δp_i can be dropped. The state vector is defined as $\mathbf{x} = [\mathbf{qp}]$. Let the ith component of the state vector be defined as $x_i = (q_i, p_i)$.

An unperturbed system with a Hamiltonian function $H^{(0)}(\mathbf{x}, t)$ will have a solution that is provided by HPF $W^{(0)}(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1)$. Consider a perturbing force that preserves the Hamiltonian structure of the system and results in a system with a Hamiltonian $H(\mathbf{x}, t) = H^{(0)}(\mathbf{x}, t) + \epsilon H^{(1)}(\mathbf{x}, t)$, where ϵ is a small parameter. Erwin Schrödinger and Lord John William Strutt Rayleigh developed a perturbation theory for the Hamiltonian where the perturbing term is linear in the Hamiltonian [10].

Because of basic existence theorems, the system $H(\mathbf{x}, t) = H^{(0)}(\mathbf{x}, t) + \epsilon H^{(1)}(\mathbf{x}, t)$ has a solution defined by a principal function $W(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1)$. Consider a Taylor series expansion of this solution, assuming ϵ can be arbitrarily small, allowing the principal function to take the form

$$W = W^{(0)} + \epsilon W^{(1)} + \epsilon^2 W^{(2)} + \cdots + \epsilon^n W^{(n)} + \cdots \tag{13}$$

where $W^{(\alpha)}$ is the α-order element of the principal function corresponding to the actual system. By definition, the full principal function satisfies

$$\mathbf{p}_0 = -\frac{\partial W}{\partial \mathbf{q}_0} \qquad \mathbf{p}_1 = \frac{\partial W}{\partial \mathbf{q}_1} \tag{14}$$

Applying this to the expanded W, define

$$\mathbf{p} = \mathbf{p}^{(0)} + \epsilon\mathbf{p}^{(1)} + \epsilon^2\mathbf{p}^{(2)} + \cdots + \epsilon^n\mathbf{p}^{(n)} + \cdots \tag{15}$$

balancing orders of ϵ leads to

$$\mathbf{p}_0^{(\alpha)} = -\frac{\partial W^{(\alpha)}}{\partial \mathbf{q}_0} \qquad \mathbf{p}_1^{(\alpha)} = \frac{\partial W^{(\alpha)}}{\partial \mathbf{q}_1} \tag{16}$$

The Hamiltonian of the system is a function of $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ and can be expressed as

$$H(\mathbf{x}, t) = H^{(0)}(\mathbf{x}, t) + \epsilon H^{(1)}(\mathbf{x}, t) = H^{(0)}(\mathbf{q}, \mathbf{p}^{(0)} + \Delta\mathbf{p}, t) + \epsilon H^{(1)}(\mathbf{q}, \mathbf{p}^{(0)} + \Delta\mathbf{p}, t) \tag{17}$$

where

$$\Delta\mathbf{p} = \sum_{\alpha=1}^{N} \epsilon^\alpha \mathbf{p}^{(\alpha)}$$

which creates an implicit recursion of higher orders of the principal function W and the Hamiltonian function H. Therefore, the Hamiltonian of the system can be expanded in a Taylor series about the principal function of the unperturbed system. Using the Einstein summation convention, this expansion is expressed as

$$H(\mathbf{x}, t) = H^{(0)}(\mathbf{x}^{(0)}, t) + \sum_{\alpha=1}^{N} \frac{1}{\alpha!} \frac{\partial^\alpha H^{(0)}}{\partial p_j \partial p_k \partial p_l \ldots}\bigg|_{\mathbf{x}^{(0)}} \Delta p_j \Delta p_k \Delta p_l \ldots + \epsilon H^{(1)}(\mathbf{x}^{(0)}, t) + \epsilon \sum_{\alpha=1}^{N} \frac{1}{\alpha!} \frac{\partial^\alpha H^{(1)}}{\partial p_j \partial p_k \partial p_l \ldots}\bigg|_{\mathbf{x}^{(0)}} \Delta p_j \Delta p_k \Delta p_l \ldots \tag{18}$$

This expression can be collected in powers of ϵ:

$$\begin{aligned} H(\mathbf{x}, t) = {} & H^{(0)}[\mathbf{x}^{(0)}(t), t] + \epsilon\left\{\frac{\partial H^{(0)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(1)} + H^{(1)}[\mathbf{x}^{(0)}(t), t]\right\} \\ & + \epsilon^2\left[\frac{\partial H^{(0)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(2)} + \frac{1}{2!}\frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(1)}\right] + \epsilon^3\left[\frac{\partial H^{(0)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(3)} + \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(2)} \right. \\ & + \frac{1}{6}\frac{\partial^3 H^{(0)}}{\partial p_j \partial p_k \partial p_l}\bigg|_{\mathbf{x}^{(0)}} p_l^{(1)} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(2)} \\ & \left. + \frac{1}{2}\frac{\partial^2 H^{(1)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(2)} p_j^{(2)}\right] + \epsilon^4\left[\frac{\partial H^{(0)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(4)} + \frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(3)} + \frac{1}{2}\frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(1)} \right. \\ & + \frac{1}{2}\frac{\partial^3 H^{(0)}}{\partial p_j \partial p_k \partial p_l}\bigg|_{\mathbf{x}^{(0)}} p_l^{(1)} p_k^{(1)} p_j^{(2)} \\ & + \frac{1}{24}\frac{\partial^4 H^{(0)}}{\partial p_j \partial p_k \partial p_l \partial p_m}\bigg|_{\mathbf{x}^{(0)}} p_m^{(1)} p_l^{(1)} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(3)} \\ & \left. + \frac{\partial^2 H^{(1)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(2)} + \frac{1}{6}\frac{\partial^3 H^{(0)}}{\partial p_j \partial p_k \partial p_l}\bigg|_{\mathbf{x}^{(0)}} p_l^{(1)} p_k^{(1)} p_j^{(1)}\right] + \cdots \end{aligned} \tag{19}$$

The action integral of the system is

$$W^{(0)} + \epsilon W^{(1)} + \cdots + \epsilon^n W^{(n)} + \cdots = \int_{t_0}^{t_1} (\mathbf{p}^{(0)} + \epsilon\mathbf{p}^{(1)} + \cdots + \epsilon^n\mathbf{p}^{(n)} + \cdots)\, \mathrm{d}\mathbf{q} - \int_{t_0}^{t_1} H(\mathbf{q}, \mathbf{p}^{(0)} +, \mathbf{p}^{(1)} + \cdots + \epsilon^n\mathbf{p}^{(n)} + \cdots, t)\, \mathrm{d}t \tag{20}$$

where the Hamiltonian is expressed by Eq. (19). Taking the variation in endpoints and time of Eq. (20) under the assumption that the trajectory satisfies the Hamiltonian equations of motion for the perturbed system yields

$$\delta W = -(\mathbf{p}_0^{(0)} + \cdots + \epsilon^n\mathbf{p}_0^{(n)} + \cdots)\delta\mathbf{q}_0 + (\mathbf{p}_1^{(0)} + \cdots + \epsilon^n\mathbf{p}_1^{(n)} + \cdots)\delta\mathbf{q}_1 + H(\mathbf{q}_0, \mathbf{p}_0^{(0)} + \cdots + \epsilon^n\mathbf{p}_0^{(n)} + \cdots, t)\delta t_0 - H(\mathbf{q}_1, \mathbf{p}_1^{(0)} + \cdots + \epsilon^n\mathbf{p}_1^{(n)} + \cdots, t)\delta t_1 \tag{21}$$

However, the variation of the principal function can also be expressed as

$$\delta W = \frac{\partial W}{\partial q_{0i}}\delta q_{0i} + \frac{\partial W}{\partial t_0}\delta t_0 + \frac{\partial W}{\partial q_{1i}}\delta q_{1i} + \frac{\partial W}{\partial t_1}\delta t_1 \tag{22}$$

Equating the δt_0 and δt_1 terms leads to

$$\begin{aligned} & \frac{\partial W^{(0)}}{\partial t_0} + \cdots + \epsilon^n\frac{\partial W^{(n)}}{\partial t_0} + \cdots - H(\mathbf{q}_0, \mathbf{p}_0^{(0)} + \cdots + \epsilon^n\mathbf{p}_0^{(n)} + \cdots, t) = 0 \\ & \frac{\partial W^{(0)}}{\partial t_1} + \cdots + \epsilon^n\frac{\partial W^{(n)}}{\partial t_1} + \cdots + H(\mathbf{q}_1, \mathbf{p}_1^{(0)} + \cdots + \epsilon^n\mathbf{p}_1^{(n)} + \cdots, t) = 0 \end{aligned} \tag{23}$$

where, by definition, the momenta at each order are

$$\begin{aligned} & \mathbf{p}_0^{(0)} = -\frac{\partial W^{(0)}}{\partial \mathbf{q}_0} \qquad \mathbf{p}_1^{(0)} = \frac{\partial W^{(0)}}{\partial \mathbf{q}_1} \qquad \cdots \qquad \cdots \\ & \mathbf{p}_0^{(n)} = -\frac{\partial W^{(n)}}{\partial \mathbf{q}_0} \qquad \mathbf{p}_1^{(n)} = \frac{\partial W^{(n)}}{\partial \mathbf{q}_1} \qquad \cdots \qquad \cdots \end{aligned} \tag{24}$$

Using the expanded Hamiltonian in Eq. (19), Eq. (23) can be rewritten as

$$\begin{aligned} & \frac{\partial W^{(0)}}{\partial t_0} - H^{(0)}(\mathbf{x}_0^{(0)}, t) + \epsilon\left[\frac{\partial W^{(1)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0j}^{(1)} - H^{(0)}(\mathbf{x}_0^{(0)}, t)\right] \\ & \quad + \epsilon^2\left[\frac{\partial W^{(2)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0j}^{(2)} - \frac{1}{2!}\frac{\partial^2 H^{(0)}}{\partial p_{0j}\partial p_{0k}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0k}^{(1)} p_{0j}^{(1)} - \frac{\partial H^{(1)}}{\partial q_{0j}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0j}^{(1)}\right] + \cdots = 0 \quad \text{and} \\ & \frac{\partial W^{(0)}}{\partial t_1} + H^{(0)}(\mathbf{x}_1^{(0)}, t) + \epsilon\left[\frac{\partial W^{(1)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1j}^{(1)} + H^{(1)}(\mathbf{x}_1^{(0)}, t)\right] + \epsilon^2\left[\frac{\partial W^{(2)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1j}^{(2)} + \frac{1}{2!}\frac{\partial^2 H^{(0)}}{\partial p_{1j}\partial p_{1k}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1k}^{(1)} p_{1j}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1j}^{(1)}\right] + \cdots = 0 \end{aligned} \tag{25}$$

From the unperturbed system, the nominal solution $W^{(0)}$ satisfies

$$\frac{\partial W^{(0)}}{\partial t_0} - H^{(0)}(\mathbf{x}_0^{(0)}, t) = 0 \qquad \frac{\partial W^{(0)}}{\partial t_1} + H^{(0)}(\mathbf{x}_1^{(0)}, t) = 0 \tag{26}$$

Hence, Eq. (25) yields the following expressions for increasing orders of ϵ:

$$\epsilon^1\colon \frac{\partial W^{(1)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0j}^{(1)} - H^{(1)}(\mathbf{x}_0^{(0)}, t) = 0$$

$$\frac{\partial W^{(1)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1j}^{(1)} + H^{(1)}(\mathbf{x}_1^{(0)}, t) = 0$$

$$\epsilon^2\colon \frac{\partial W^{(2)}}{\partial t_0} - \frac{\partial H^{(0)}}{\partial p_{0j}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0j}^{(2)} - \frac{1}{2!}\frac{\partial^2 H^{(0)}}{\partial p_{0j}\partial p_{0k}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0k}^{(1)} p_{0j}^{(1)} - \frac{\partial H^{(1)}}{\partial p_{0j}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0j}^{(1)} = 0$$

$$\frac{\partial W^{(2)}}{\partial t_1} + \frac{\partial H^{(0)}}{\partial p_{1j}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1j}^{(2)} + \frac{1}{2!}\frac{\partial^2 H^{(0)}}{\partial p_{1j}\partial p_{1k}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1k}^{(1)} p_{1j}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1j}^{(1)} = 0 \qquad \cdots \qquad \cdots \tag{27}$$

Following Hamilton, it is noted that the total derivative of $W^{(\alpha)}$ with respect to t_0 and t_1 can be expressed as [11]

$$\frac{\mathrm{d}W^{(\alpha)}}{\mathrm{d}t_0} = \frac{\partial W^{(\alpha)}}{\partial t_0} + \frac{\partial W^{(\alpha)}}{\partial q_{0i}}\frac{\mathrm{d}q_{0i}}{\mathrm{d}t_0} \qquad \frac{\mathrm{d}W^{(\alpha)}}{\mathrm{d}t_1} = \frac{\partial W^{(\alpha)}}{\partial t_1} + \frac{\partial W^{(\alpha)}}{\partial q_{1i}}\frac{\mathrm{d}q_{1i}}{\mathrm{d}t_1} \tag{28}$$

where $\mathbf{q}_0$ and $\mathbf{q}_1$ lie along the nominal trajectory. By definition, $\mathrm{d}\mathbf{q}_0/\mathrm{d}t_0 = (\partial H/\partial \mathbf{p}_0)|_{\mathbf{x}_0^{(0)}}$ and $\mathrm{d}\mathbf{q}_1/\mathrm{d}t_1 = (\partial H/\partial \mathbf{p}_1)|_{\mathbf{x}_1^{(0)}}$, since the expansion is about this nominal solution. Also, from Eq. (16), $\partial W^{(\alpha)}/\partial \mathbf{q}_0 = -\mathbf{p}_0^{(\alpha)}$ and $\partial W^{(\alpha)}/\partial \mathbf{q}_1 = \mathbf{p}_1^{(\alpha)}$; therefore, Eq. (28) becomes

$$\frac{\mathrm{d}W^{(\alpha)}}{\mathrm{d}t_0} = \frac{\partial W^{(\alpha)}}{\partial t_0} - p_{0i}^{(\alpha)}\frac{\partial H}{\partial p_{0i}}\bigg|_{\mathbf{x}_0^{(0)}}$$

$$\frac{\mathrm{d}W^{(\alpha)}}{\mathrm{d}t_1} = \frac{\partial W^{(\alpha)}}{\partial t_1} + p_{1i}^{(\alpha)}\frac{\partial H}{\partial p_{1i}}\bigg|_{\mathbf{x}_1^{(0)}} \tag{29}$$

where $\partial H/\partial \mathbf{p}_0|_{\mathbf{x}_0^{(0)}}$ is a function of the initial nominal state $(\mathbf{q}_0, \mathbf{p}_0^{(0)})$, and $\partial H/\partial \mathbf{p}_1|_{\mathbf{x}_0^{(0)}}$ is a function of the final nominal state $(\mathbf{q}_1, \mathbf{p}_1^{(0)})$. Therefore, substituting the right-hand side of Eq. (29) into Eq. (27) leads to the total time derivative expressions for $W^{(\alpha)}$, which can be written as

$$\epsilon^1\colon \frac{\mathrm{d}W^{(1)}}{\mathrm{d}t_0} - H^{(1)}(\mathbf{x}_0^{(0)}, t) = 0 \qquad \frac{\mathrm{d}W^{(1)}}{\mathrm{d}t_1} + H^{(1)}(\mathbf{x}_{1i}^{(0)}, t) = 0$$

$$\epsilon^2\colon \frac{\mathrm{d}W^{(2)}}{\mathrm{d}t_0} - \frac{1}{2!}\frac{\partial^2 H^{(0)}}{\partial p_{0j}\partial p_{0k}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0k}^{(1)} p_{0j}^{(1)} - \frac{\partial H^{(1)}}{\partial p_{0j}}\bigg|_{\mathbf{x}_0^{(0)}} p_{0j}^{(1)} = 0$$

$$\frac{\mathrm{d}W^{(2)}}{\mathrm{d}t_1} + \frac{1}{2!}\frac{\partial^2 H^{(0)}}{\partial p_{1j}\partial p_{1k}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1k}^{(1)} p_{1j}^{(1)} + \frac{\partial H^{(1)}}{\partial p_{1j}}\bigg|_{\mathbf{x}_1^{(0)}} p_{1j}^{(1)} = 0$$

$$\cdots$$

$$\cdots$$

$$\epsilon^n\colon \frac{\mathrm{d}W^{(n)}}{\mathrm{d}t_0} = F_0^{(n)}(\mathbf{x}_0^{(0)}, W^{(0)}, W^{(1)}, \cdots, W^{(n-1)})$$

$$\frac{\mathrm{d}W^{(n)}}{\mathrm{d}t_1} = F_1^{(n)}(\mathbf{x}_1^{(0)}, W^{(0)}, W^{(1)}, \ldots, W^{(n-1)})$$

$$\cdots$$

$$\cdots \tag{30}$$

Note that each differential equation is a function of only lower-order terms, which are already solved for as a function of time. Thus, each of these differential equations can be solved for as a quadrature in time. As can be seen, each order $W^{(\alpha)}$ apparently needs to satisfy two equations simultaneously. Hence,

$$\epsilon^1\colon W^{(1)} = \int_{t_1}^{t_0} H^{(1)}(\mathbf{x}^{(0)}(t), t)\,\mathrm{d}t$$

$$W^{(1)} = -\int_{t_0}^{t_1} H^{(1)}(\mathbf{x}^{(0)}(t), t)\,\mathrm{d}t$$

$$\epsilon^2\colon W^{(2)} = \int_{t_1}^{t_0}\left[\frac{1}{2}\frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(1)}\right]\mathrm{d}t$$

$$W^{(2)} = -\int_{t_0}^{t_1}\left[\frac{1}{2}\frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(1)}\right]\mathrm{d}t$$

$$\cdots$$

$$\cdots$$

$$\epsilon^n\colon W^{(n)} = -\int_{t_1}^{t_0} F^{(n)}(\mathbf{x}^{(0)}(t), t, W^{(0)}, \ldots, W^{(n-1)})\,\mathrm{d}t$$

$$W^{(n)} = \int_{t_0}^{t_1} F^{(n)}(\mathbf{x}^{(0)}(t), t, W^{(0)}, \ldots, W^{(n-1)})\,\mathrm{d}t$$

$$\cdots$$

$$\cdots \tag{31}$$

However, these two simultaneous equations are one and the same, as can be seen by reversing the limits of integration. Hence, at each order, $W^{(\alpha)}$ is defined by only one equation, which is

$$\epsilon^1\colon W^{(1)} = -\int_{t_0}^{t_1} H^{(1)}(\mathbf{x}^{(0)}(t), t)\,\mathrm{d}t$$

$$\epsilon^2\colon W^{(2)} = -\int_{t_0}^{t_1}\left[\frac{1}{2}\frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(1)}\right]\mathrm{d}t$$

$$\cdots$$

$$\cdots$$

$$\epsilon^n\colon W^{(n)} = \int_{t_0}^{t_1} F^{(n)}[\mathbf{x}^{(0)}(t), t, W^{(0)}, \ldots, W^{(n-1)}]\,\mathrm{d}t \tag{32}$$

These equations allow one to recursively solve for the principal function of the full system once the nominal solution $W^{(0)}$ is known, since at each order, $W^{(\alpha)}$ depends on $W^{(0)}, W^{(1)}, \ldots, W^{(\alpha-1)}$. To solve the 2BVP, one just needs to take the partial derivatives of W with respect to q_{0i} and q_{1i}. However, this requires the knowledge of an analytical expression for the state vector of the nominal system, which is not always possible.

IV. Solving Two-Point Boundary Value Problems

Consider the 2BVP that connects two points in phase space, $[\mathbf{q}_0(t_0), \mathbf{p}_0(t_0), t_0]$ and $[\mathbf{q}_1(t_1), \mathbf{p}_1(t_1), t_1]$. Given the general coordinates $\mathbf{q}_0(t_0)$ and $\mathbf{q}_1(t_1)$, the required momenta $\mathbf{p}_0(t_0)$ and $\mathbf{p}_1(t_1)$ that connects the two points needs to be solved.

A. Two-Point Boundary Value Solutions

Recall that the principal function solves the 2BVP by taking its partial derivative with respect to the endpoint coordinates

$$\mathbf{p}_0 = -\frac{\partial W}{\partial \mathbf{q}_0}(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1) \qquad \mathbf{p}_1 = \frac{\partial W}{\partial \mathbf{q}_1}(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1) \tag{33}$$

At each order, the principal function and momenta are related by

$$\mathbf{p}_0^{(\alpha)} = -\frac{\partial W^{(\alpha)}}{\partial \mathbf{q}_0} \qquad \mathbf{p}_1^{(\alpha)} = \frac{\partial W^{(\alpha)}}{\partial \mathbf{q}_1} \tag{34}$$

Thus, if W is computed to order N, the solution of the 2BVP to that order can be approximated using

$$W = \sum_{\alpha=1}^{N} \epsilon^{\alpha} W^{(\alpha)}$$

to find

$$\mathbf{p} = \sum_{\alpha=1}^{N} \epsilon^{\alpha} \mathbf{p}^{(\alpha)}$$

Therefore,

$$\epsilon^1\colon \mathbf{p}_0^{(1)} = -\frac{\partial W^{(1)}}{\partial \mathbf{q}_0} = \frac{\partial}{\partial \mathbf{q}_0}\int_{t_0}^{t_1} H^{(1)}[\mathbf{x}^{(0)}(t), t]\,\mathrm{d}t$$

$$\mathbf{p}_1^{(1)} = \frac{\partial W^{(1)}}{\partial \mathbf{q}_1} = -\frac{\partial}{\partial \mathbf{q}_1}\int_{t_0}^{t_1} H^{(1)}[\mathbf{x}^{(0)}(t), t]\,\mathrm{d}t$$

$$\epsilon^2\colon \mathbf{p}_0^{(2)} = -\frac{\partial W^{(2)}}{\partial \mathbf{q}_0} = \frac{\partial}{\partial \mathbf{q}_0}\int_{t_0}^{t_1}\left[\frac{1}{2}\frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(1)}\right]\mathrm{d}t$$

$$\mathbf{p}_1^{(2)} = \frac{\partial W^{(2)}}{\partial \mathbf{q}_1} = -\frac{\partial}{\partial \mathbf{q}_1}\int_{t_0}^{t_1}\left[\frac{1}{2}\frac{\partial^2 H^{(0)}}{\partial p_j \partial p_k}\bigg|_{\mathbf{x}^{(0)}} p_k^{(1)} p_j^{(1)} + \frac{\partial H^{(1)}}{\partial p_j}\bigg|_{\mathbf{x}^{(0)}} p_j^{(1)}\right]\mathrm{d}t$$

$$\cdots$$

$$\cdots$$

$$\epsilon^n\colon \mathbf{p}_0^{(n)} = -\frac{\partial W^{(n)}}{\partial \mathbf{q}_0} = -\frac{\partial}{\partial \mathbf{q}_0}\int_{t_0}^{t_1} F^{(n)}[\mathbf{x}^{(0)}(t), t, W^{(0)}, \ldots, W^{(n-1)}]\,\mathrm{d}t$$

$$\mathbf{p}_1^{(n)} = \frac{\partial W^{(n)}}{\partial \mathbf{q}_1} = \frac{\partial}{\partial \mathbf{q}_1}\int_{t_0}^{t_1} F^{(n)}[\mathbf{x}^{(0)}(t), t, W^{(0)}, \ldots, W^{(n-1)}]\,\mathrm{d}t$$

$$\cdots$$

$$\cdots \tag{35}$$

The general form of Eq. (35) is

$$\mathbf{p}_0 = -\int_{t_0}^{t_1} \frac{\partial F^{(\alpha)}}{\partial \mathbf{q}_0}\,\mathrm{d}t \qquad \mathbf{p}_1 = \int_{t_0}^{t_1} \frac{\partial F^{(\alpha)}}{\partial \mathbf{q}_1}\,\mathrm{d}t \tag{36}$$

where

$$\frac{\partial F^{(\alpha)}}{\partial q_{0i}} = \frac{\partial F^{(\alpha)}}{\partial x_j^{(0)}}\frac{\partial_b x_j^{(0)}}{\partial_b q_{0i}} \tag{37}$$

$$\frac{\partial F^{(\alpha)}}{\partial q_{1i}} = \frac{\partial F^{(\alpha)}}{\partial x_j^{(0)}}\frac{\partial_b x_j^{(0)}}{\partial_b q_{1i}} \tag{38}$$

since $F^{(\alpha)} = F^{(\alpha)}[t, \mathbf{x}^{(0)}(t, \mathbf{x}_0, t_0)]$

Figure 1 illustrates the problem in the presence of a perturbation. For the variation of $x^{(0)}$ with respect to $\mathbf{q}_0$, there are constraints that affect the motion of the system in phase space. Since the 2BVP is being solved, the endpoint generalized coordinates are fixed, even if the path in phase space of the dynamics differs from the nominal. Therefore, any variation of the initial generalized coordinate $\mathbf{q}_0$ has to result in the final endpoint coordinate being fixed; hence, any variation of the final coordinate will be equal to zero, $\delta\mathbf{q}_1 = 0$, and this fact imposes a constraint on the dynamics of the system. Therefore, under these circumstances, the nominal solution $\mathbf{x}^{(0)}$ is a function of the endpoint constraint and the proper chain rule has to be applied.

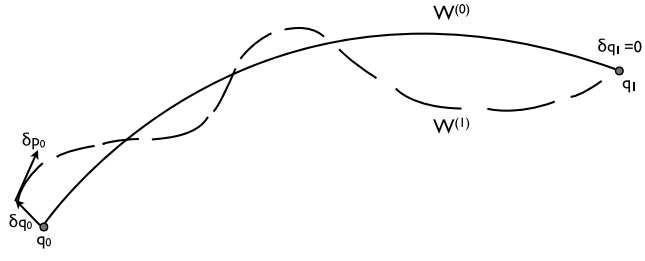

Fig. 1 Illustration of constraint endpoint.

The subscript b in the partial derivative corresponds to the constraint case that solves the boundary value problem using the nominal solution. This partial derivative consists of direct and indirect parts. The direct part comes from the state transition matrix, while the indirect part is the necessary variation to ensure that $\delta\mathbf{q}(t_1) = 0$.

B. Solving the Constrained Partial Derivative

To solve the aforementioned constrained partial derivative, consider the state vector of the nominal solution at some arbitrary time $\tau\epsilon[t_0, t_1]$:

$$\mathbf{x}^{(0)}(\tau) = \mathbf{x}^{(0)}[\tau, \mathbf{q}^{(0)}(t), \mathbf{p}^{(0)}(t), t] \tag{39}$$

The state vector is dependent on the state at a time $t\epsilon[t_0, \tau]$. Consider a variation to the state vector at time t:

$$\mathbf{x}(t) = \mathbf{x}^{(0)}(t) + \delta\mathbf{x}(t) \tag{40}$$

The state vector at time τ due to variation in the state at t is therefore

$$\mathbf{x}^{(0)}[\tau, \mathbf{q}^{(0)}(t) + \delta\mathbf{q}(t), \mathbf{p}^{(0)}(t) + \delta\mathbf{p}(t), t] = \mathbf{x}(\tau) + \delta\mathbf{x}(\tau) \tag{41}$$

The variation on generalized coordinates at time τ is

$$\mathbf{q}^{(0)}(\tau) + \delta\mathbf{q}(\tau) = \mathbf{q}^{(0)}(\tau) + \Phi_{j\alpha}\delta q_\alpha(t) + \Phi_{j(n+\alpha)}\delta p_\alpha(t) + \frac{1}{2!}[\Phi_{j\alpha\beta}\delta q_\alpha(t)\delta q_\beta(t) + \Phi_{j(n+\alpha)\beta}\delta p_\alpha(t)\delta q_\beta(t) + \Phi_{j\alpha(n+\beta)}\delta q_\alpha(t)\delta p_\beta(t) + \Phi_{j(n+\alpha)(n+\beta)}\delta p_\alpha(t)\delta p_\beta(t)] + \cdots \tag{42}$$

where $\Phi_{ij}(\tau, t)$ is the state transition matrix and $\Phi_{ijkl\ldots}(\tau, t)$ are the higher-order partials of the state transition matrix or state transition tensors. Note that these tensors can be solved for from the nominal integrable solution. The dimension of the state vector is $2n$.

For this particular 2BVP, $\mathbf{q}(t_1) = \mathbf{q}_1$ is fixed and the variation $\delta\mathbf{q}(\tau)$ will vanish when $\tau = t_1$; hence, $\delta\mathbf{q}_1 = 0$. This condition imposes an n-dimensional constraint on the system, allowing us to solve for the other n variables.

Assume that momenta can be expressed as a function of coordinates only, $\mathbf{p}(t) = f(\mathbf{q})$. This assumption allows us to express the variation of the momenta as a power series of the form

$$\delta\mathbf{p}(t) = C_{i\gamma}\delta q_\gamma(t) + C_{i\gamma\psi}\delta q_\gamma(t)\delta q_\psi(t) + \cdots \tag{43}$$

where C is coefficient tensors to be determined by boundary conditions. Substituting Eq. (43) into Eq. (42) and combining terms yields

$$\delta\mathbf{q}(\tau) = [\Phi_{j\gamma} + \Phi_{j(n+\alpha)}C_{\alpha\gamma}]\delta q_\gamma(t) + \tfrac{1}{2}[\Phi_{j\gamma\psi} + \Phi_{j(n+\alpha)\psi}C_{\alpha\gamma} + \Phi_{j\gamma(n+\beta)}C_{\beta\psi} + \Phi_{j(n+\alpha)(n+\beta)}C_{\alpha\gamma}C_{\beta\delta} + 2\Phi_{j(n+\alpha)}C_{\alpha\gamma\psi}]\delta q_\gamma(t)\delta q_\psi(t) + \cdots \tag{44}$$

Therefore, $\delta\mathbf{q}(t_1) = 0$ and, using this boundary condition, the coefficient tensors in Eq. (44) can be solved at each order

$$C_{\alpha\gamma}(t_1, t) = -\Phi^{-1}_{j(n+\alpha)}(t_1, t)\Phi_{j\gamma}(t_1, t)$$

$$C_{\alpha\gamma\psi}(t_1, t) = -\tfrac{1}{2}\Phi^{-1}_{j(n+\alpha)}(t_1, t)[\Phi_{j\gamma\psi}(t_1, t) + 2\Phi_{j(n+\alpha)\psi}(t_1, t)C_{\alpha\gamma} + \Phi_{j(n+\alpha)(n+\beta)}(t_1, t)C_{\alpha\gamma}(t_1, t)C_{\beta\psi}(t_1, t)]$$

$$\cdots \tag{45}$$

The tensors are functions of the state transition matrix and state transition tensors of the nominal solution from t to t_1. To solve these, it is necessary for unique inverses to exist, which are not always guaranteed. If these tensors are singular, they correspond to the

existence of multiple solutions to the nominal 2BVP. Discussions on how to solve the problem in the presence of singularities was explored by Guibout [8]. Hence, the constrained partial derivatives can now be solved:

$$\frac{\partial_b x_j^{(0)}}{\partial_b q_{0\gamma}}(\tau,t) = \Phi_{j\gamma}(\tau,t) + \Phi_{j(n+\alpha)}(\tau,t)C_{\alpha\gamma}(t_1,t)$$

$$\frac{\partial_b^2 x_j^{(0)}}{\partial_b q_{0\gamma}\partial_b q_{0\psi}}(\tau,t) = \frac{1}{2}[\Phi_{j\gamma\psi}(\tau,t) + \Phi_{j(n+\alpha)\psi}(\tau,t)C_{\alpha\gamma}(t_1,t) + \Phi_{j\gamma(n+\beta)}(\tau,t)C_{\beta\psi}(t_1,t) + \Phi_{j(n+\alpha)(n+\beta)}(\tau,t)C_{\alpha\gamma}(t_1,t)C_{\beta\delta}(t_1,t) + 2\Phi_{j(n+\alpha)}(\tau,t)C_{\alpha\gamma\psi}(t_1,t)]\cdots \tag{46}$$

Note that the solution at each order can be obtained by the knowledge of the nominal solution $\mathbf{x}^{(0)}$, since $W^{(n)} = W^{(n)}(W^{(0)})$. Thus, the expressions in Eq. (35) can be solved using quadratures.

From a practical standpoint, the state transition matrix and state transition tensors can also be obtained numerically. Park and Scheeres derived a scheme to obtain numerical expressions for these quantities [12].

C. Numerical Error Analysis

Carrying out a double integral by means of quadratures inevitably leads to numerical errors in calculations. To obtain highly accurate results in a perturbed system, it is important to understand how these numerical errors may change the obtained solutions.

The calculations for the integrals involved in the perturbed solutions are carried out by simple quadratures or using the midpoint rule. For an integrand $f(t)$ integrating from t_0 to t_1, the integral can be expressed as

$$\int_{t_0}^{t_1} F(t)\,\mathrm{d}t = \sum_{\alpha=1}^{N}\left[\Delta F_\alpha \frac{t_1-t_0}{N}\right] + \frac{(t_1-t_0)^3}{24N}\frac{\mathrm{d}^2F}{\mathrm{d}t^2}\bigg|_{t=\xi} \tag{47}$$

where $\xi\epsilon[t_0,t_1]$. The difference between the calculated first-order theory and the true solution i is

$$p_0^{(1)*} = p_0^{(1)} + \Delta p_0^{(1)} \tag{48}$$

where $p_0^{(1)*}$ is the true first-order solution

$$p_0^{(1)*} = \int_{t_0}^{t_1} \mathrm{d}p_0^{(1)*} \tag{49}$$

where $p_0^{(1)}$ is the numerically obtained first-order solution

$$p_0^{(1)} = \sum_{i=1}^{N} p_{0i}^{(1)} \tag{50}$$

Choosing $\xi = t_0$ yields

$$\delta p_0^{(1)} = \frac{(t_1-t_0)^3}{24N^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[-\frac{\partial H^{(1)}}{\partial q_0}\right]\bigg|_{t=t_0} = \frac{(t_1-t_0)^3}{24N^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{\partial^2 H^{(1)}}{\partial q_0^2}\frac{\mathrm{d}q_0}{\mathrm{d}t_0}\right]\bigg|_{t=t_0} \tag{51}$$

Recall that $\mathrm{d}q_0/\mathrm{d}t_0 = -p_0$; therefore,

$$\Delta p_0^{(1)} = -\frac{(t_1-t_0)^3}{24N^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial^2 H^{(1)}}{\partial q_0^2}p_0\right]\bigg|_{t=t_0} \tag{52}$$

Therefore,

$$\Delta p_0^{(1)} = \frac{(t_1-t_0)^3}{24N^2}\left[\frac{\partial^3 H^{(1)}}{\partial q_0^3}(p_0)^2 + \frac{\partial^2 H^{(1)}}{\partial q_0^2}\frac{\partial p_0}{\partial q_0}p_0\right] \tag{53}$$

The first-order error will depend on the time of flight, the integration time step, the perturbing function, and the nominal initial velocity of the orbiting particle. For a given 2BVP, the numerical error will be a function of the chosen step size for that particular numerical integration:

$$\Delta p_0^{(1)} \propto \frac{1}{N^2} \tag{54}$$

Similarly, for the second order, the true solution is

$$p_0^{(2)*} = \int_{t_0}^{t_1} \mathrm{d}p_0^{(2)*} \tag{55}$$

and the calculated solution is

$$p_0^{(2)} = \sum_{\alpha=1}^{N} p_{0\alpha}^{(2)} \tag{56}$$

The numerical second-order error is

$$\Delta p_0^{(2)} = -\frac{(t_1-t_0)^3}{24N^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{\partial p^{(1)*}}{\partial q_0}p^{(1)*}\right]\bigg|_{t=t_0} \tag{57}$$

$$\Delta p_0^{(2)} = -\frac{(t_1-t_0)^3}{24N^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\left(-\int_t^{t_1}\frac{\partial^2 H^{(1)}}{\partial q_0^2}\,\mathrm{d}t\right)\times\left(-\int_t^{t_1}\frac{\partial H^{(1)}}{\partial q_0}\,\mathrm{d}t\right)\right]\bigg|_{t=t_0} \tag{58}$$

$$\Delta p_0^{(2)} = -\frac{(t_1-t_0)^3}{24N^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{\partial^2 H^{(1)}}{\partial q_0^2}p_0^{(1)*} - \frac{\partial p_0^{(1)*}}{\partial q_0}\frac{\partial H^{(1)}}{\partial q_0}\right]\bigg|_{t=t_0} \tag{59}$$

Hence,

$$\Delta p_0^{(2)} = -\frac{(t_1-t_0)^3}{24N^2}\left[\left(\frac{\partial^3 H^{(1)}}{\partial q_0^3}p_0^{(1)*} + \frac{\partial p_0^{(1)*}}{\partial q_0}\frac{\partial^2 H^{(1)}}{\partial q_0^2}\right)p_0 + 2\frac{\partial^2 H^{(1)}}{\partial q_0^2}\frac{\partial H^{(1)}}{\partial q_0}\right] \tag{60}$$

Consider now the difference between the true second-order and the calculated second-order solutions:

$$p_0^{(2)*} = p_0^{(2)} + \Delta p_0^{(2)} = \sum_{\alpha=1}^{N} p_{0\alpha}^{(2)} + \Delta p_0^{(2)} \tag{61}$$

where $p_{0\alpha}^{(2)}$ is the numerically computed solution that depends on the numerically computed first-order solution $p_\alpha^{(1)}$. Therefore,

$$\sum_{\alpha=1}^{N} p_{0\alpha}^{(2)}(p_\alpha^{(1)}) = \sum_{\alpha=1}^{N} p_\alpha^{(2)}(p_\alpha^{(1)*} - \Delta p_\alpha^{(1)}) = \sum_{\alpha=1}^{N} p_{0\alpha}^{(2)}(p_i^{(1)*}) - \sum_{\alpha=1}^{N}\frac{\partial p_\alpha^{(2)}}{\partial p_\alpha^{(1)*}}\Delta p_\alpha^{(1)} - \sum_{\alpha=1}^{N}\frac{\partial^2 p_0^{(2)}}{\partial (p_\alpha^{(1)*})^2}(\Delta p_\alpha^{(1)})^2 \tag{62}$$

where

$$p_{0\alpha}^{(2)} = \frac{\partial(p_\alpha^{(1)*} - \Delta p_\alpha^{(1)})}{\partial q_0}(p_\alpha^{(1)*} - \Delta p_\alpha^{(1)}) \tag{63}$$

Expanding the preceding equation,

$$p_{0\alpha}^{(2)} = \frac{\partial p_\alpha^{(1)*}}{\partial q_0}p_\alpha^{(1)*} - \frac{\partial p_\alpha^{(1)*}}{\partial q_0}\Delta p_\alpha^{(1)*} - \frac{\partial \Delta p_\alpha^{(1)}}{\partial q_0}p_\alpha^{(1)*} + \frac{\partial \Delta p_\alpha^{(1)}}{\partial q_0}\Delta p_\alpha^{(1)} \tag{64}$$

Therefore,

$$\frac{\partial p_{0\alpha}^{(2)}}{\partial p_\alpha^{(1)*}} = \frac{\partial p_\alpha^{(1)*}}{\partial q_0} = \int_{t_0}^{t_1}\frac{\partial^2 H^{(1)}}{\partial q_0^2}\,\mathrm{d}t \tag{65}$$

which leads to

$$\Delta p_0^{(2)} = p_0^{(2)*} - \sum_{\alpha=1}^{N} \Delta p_{0\alpha}^{(2)}(p_\alpha^{(1)*}) = \delta p_0^{(2)} - \sum_{\alpha=1}^{N} \frac{\partial p_\alpha^{(1)*}}{\partial q_0} \delta p_\alpha^{(1)} \quad (66)$$

The second-order error depends on the numerical error from the integration routine, which as with the first order is proportional to $1/N^2$. However, there is additional numerical error caused by the first-order integration numerical error, which is carried over. Therefore, it is necessary to obtain the first-order numerical solution as accurately as possible in order to obtain an improved second-order solution.

D. Example: One-Dimensional Motion

To illustrate the described perturbation theory, a simple example is presented in the following. Consider the nominal system to be a particle moving along a straight line with constant velocity. The system has the set of initial conditions (x_0, v_0, t_0) and final conditions (x_1, v_1, t_1), and its solution is described by the following equations:

$$H = \frac{1}{2}v^2 \qquad v_1 = \frac{x_2 - x_1}{t_2 - t_1} \qquad v_2 = \frac{x_2 - x_1}{t_2 - t_1}$$
$$x_2 = x_1 + v_1(t_2 - t_1) \qquad x_1 = x_2 - v_2(t_2 - t_1) \quad (67)$$

The principal function for this nominal system is

$$W = \frac{1}{2}\frac{(x_1 - x_0)^2}{t_1 - t_0} \quad (68)$$

Letting the system undergo a perturbation results in the Hamiltonian $H = \frac{1}{2}v^2 - \epsilon a x$. The system and its solutions are now described by

$$H = H^{(0)} + \epsilon H^{(1)} = \frac{1}{2}v^2 - \epsilon a x$$
$$v_0 = v_0^{(0)} + \epsilon v_0^{(1)} = \frac{x_1 - x_0}{t_1 - t_0} - \epsilon\frac{1}{2}a(t_1 - t_0)$$
$$v_2 = v_1^{(0)} + \epsilon v_1^{(1)} = \frac{x_1 - x_0}{t_1 - t_0} + \epsilon\frac{1}{2}a(t_1 - t_0)$$
$$x_1 = x_0 + v_0^{(0)}(t_1 - t_0) + \epsilon\frac{1}{2}a(t_1 - t_0)^2$$
$$x_0 = x_1 - v_1^{(0)}(t_1 - t_0) - \epsilon\frac{1}{2}a(t_1 - t_0)^2 \quad (69)$$

The principal function for the solution of this system is now

$$W = \frac{1}{2}\frac{(x_1 - x_0)^2}{t_1 - t_0} + \epsilon\frac{1}{2}a(x_0 + x_1)(t_1 - t_0) - \epsilon^2\frac{1}{24}a^2(t_1 - t_0)^3 \quad (70)$$

It can easily be shown that this is the principal function for the solution of the system

$$-\frac{\partial W}{\partial x_0} = v_0 = \frac{x_1 - x_0}{t_1 - t_0} - \epsilon\frac{1}{2}a(t_1 - t_0)$$
$$\frac{\partial W}{\partial x_1} = v_1 = \frac{x_1 - x_0}{t_1 - t_0} + \epsilon\frac{1}{2}a(t_1 - t_0) \quad (71)$$

Application of the perturbation theory leads to

$$W^{(1)} = a\int_{t_0}^{t_1} x(t)\,\mathrm{d}t \qquad W^{(2)} = -\frac{1}{8}a^2\int_{t_0}^{t_1}(t - t_0)^2\,\mathrm{d}t \quad (72)$$

where the coordinate of the unperturbed system is

$$x(t) = x_0 + [(x_1 - x_0)/(t_1 - t_0)](t - t_0)$$

Therefore, integrating these equations yield

$$W^{(1)} = \frac{1}{2}a(x_0 + x_1)(t_1 - t_0) \qquad W^{(2)} = -\frac{1}{24}a^2(t_1 - t_0)^3 \quad (73)$$

and $W^{(0)} + \epsilon W^{(1)} + \epsilon^2 W^{(2)} = W$. To solve the 2BVP, it suffices to differentiate W with respect to x_0 and x_1.

In this example, it is possible to find an analytical solution for the principal function and then differentiate it to solve the 2BVP. The theory described in the previous section allows us to directly solve for the velocities at the endpoints without solving for the principal function by the following:

$$v_0^{(1)} = -a\int_{t_0}^{t_1}\frac{\partial x}{\partial x_0}\,\mathrm{d}t = -a\int_{t_0}^{t_1}\left[1 - \frac{t - t_0}{t_1 - t_0}\right]\mathrm{d}t = -\frac{1}{2}a(t_1 - t_0)$$
$$v_1^{(1)} = a\int_{t_0}^{t_1}\frac{\partial x}{\partial x_1}\,\mathrm{d}t = a\int_{t_0}^{t_1}\left[\frac{t - t_0}{t_1 - t_0}\right]\mathrm{d}t = \frac{1}{2}a(t_1 - t_0) \quad (74)$$

Note that $\partial W^{(2)}/\partial x_0 = \partial W^{(2)}/\partial x_1 = 0$.

V. Perturbation Analysis Examples

This section shows the validity of the theory by applying it in detail to the Keplerian two-body problem.

A. Two-Body Problem

In the two-body problem, two bodies with respective masses M_1 and M_2 orbit each other with a gravitational parameter $\mu = G(M_1 + M_2)$. The center of mass of the system is stationary, and the relative position follows the equation of motion

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\partial U_0}{\partial \mathbf{r}}$$

where $U = U(\mathbf{r})$ is the force potential. The two-body problem is a Hamiltonian dynamical system where the Hamiltonian is defined as

$$H = \frac{1}{2}\mathbf{p}\cdot\mathbf{p} - \frac{\mu}{|\mathbf{q}|} \quad (75)$$

The canonical variables $\mathbf{q}$ and $\mathbf{p}$ have been chosen to be the position $\mathbf{r}$ and the velocity $\mathbf{v}$, respectively.

B. Keplerian Two-Point Boundary Value Problem Solutions

Consider a spacecraft on a circular orbit of radius r_0. The local circular velocity is $v_0 = \sqrt{\mu_1/r_0}$, where μ_1 is the gravitational parameter of the central body. Consider an orbit transfer of the spacecraft to a circular orbit of radius r_1: the Hohmann transfer will be the optimal two-impulse maneuver for a Keplerian orbit. The impulsive changes in the initial and final velocities can be obtained as [13]

$$\Delta v_0 = \sqrt{\frac{2\mu_1}{r_0} - \frac{2\mu_1}{r_0 + r_1}} - \sqrt{\frac{\mu_1}{r_0}}$$
$$\Delta v_1 = \sqrt{\frac{\mu_1}{r_1}} - \sqrt{\frac{2\mu_1}{r_1} - \frac{2\mu_1}{r_0 + r_1}} \quad (76)$$

where

$$r_0 = |\mathbf{r}_0| \qquad r_1 = |\mathbf{r}_1| \quad (77)$$

The Hohmann transfer can be viewed as a 2BVP solution where the spacecraft travels from $(\mathbf{r}_0, \mathbf{v}_0)$ to $(\mathbf{r}_1, \mathbf{v}_1)$ in transfer time $t_1 - t_0 = T_H$, defined by an orbit with a semimajor axis a_H expressed as [13]

$$a_H = \frac{r_0 + r_1}{2} \qquad T_H = \pi\sqrt{\frac{a_H^3}{\mu_1}} = \pi\sqrt{\frac{(r_0 + r_1)^3}{8\mu_1}} \quad (78)$$

The initial and final Δv for the Hohmann transfer assume a Keplerian transfer orbit, therefore neglecting perturbing forces. If a perturbing force potential is present, one can use the previously described theory to solve the perturbed Hohmann transfer. To obtain the correction to the initial impulse, we need the solution to the nominal (Keplerian) problem, which can be obtained by numerical integration of the two-body equations of motion.

C. Lambert's Problem

For the nominal two-body problem, the Hohmann transfer between two circular orbits is the optimal two-impulse maneuver. However, this transfer is restricted to a specific target point (180 deg change in true anomaly) and a fixed transfer time; therefore, it offers poor flexibility for finding alternative transfers.

If a different transfer time or transfer angle is desired, one must instead solve Lambert's problem to obtain a transfer orbit and determine the required change in initial and final velocities. There is no closed-form solution to Lambert's problem, as it involves solving Kepler's equation, and it must be obtained through iterative methods.

A novel approach to obtaining solutions to Lambert's problem and several algorithms on how to obtain the transfer orbit can be found in the report by Jordan [14]. The formulas for the required initial and final velocities that would satisfy the solution to a given Lambert's problem can be found by taking the partial derivative of HPF. Peñagaricano and Scheeres [15] introduced this function for the Keplerian two-body problem, and a detailed derivation can be found in Appendix B:

$$W(\mathbf{r}_0,\mathbf{r}_1,t_0,t_1) = \begin{cases} \sqrt{\mu a}[(\alpha+\sin\alpha)-(\beta+\sin\beta)]+\frac{\mu}{2a}(t_1-t_0) & E<0 \\ \sqrt{8\mu}\left[\sqrt{\frac{s}{2}}-\sqrt{\frac{s-c}{2}}\right] & E=0 \\ \sqrt{-\mu a}[(\delta+\sinh\delta)-(\gamma+\sinh\gamma)]+\frac{\mu}{-2a}(t_1-t_0) & E>0 \end{cases} \tag{79}$$

where

$$\sin\frac{\alpha}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|+|\mathbf{r}_0+\mathbf{r}_1|}{4a}}$$

$$\sin\frac{\beta}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|-|\mathbf{r}_0+\mathbf{r}_1|}{4a}}$$

$$\sinh\frac{\gamma}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|+|\mathbf{r}_0+\mathbf{r}_1|}{-4a}}$$

$$\sinh\frac{\delta}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|-|\mathbf{r}_0+\mathbf{r}_1|}{-4a}}$$

$$s=\frac{|\mathbf{r}_0|+|\mathbf{r}_1|+|\mathbf{r}_0+\mathbf{r}_1|}{2}$$

$$c=|\mathbf{r}_0+\mathbf{r}_1|$$

and the semimajor axis a is related to the total energy/Hamiltonian of the system by

$$a=-\frac{\mu}{2E}$$

This principal function was then used to solve the targeting problem by differentiating the principal function with respect to the two position vectors $\mathbf{r}_0$ at t_0 and $\mathbf{r}_1$ at t_1:

$$\mathbf{v}_0=(B-A)\frac{\mathbf{r}_0}{|\mathbf{r}_0|}+(B+A)\frac{\mathbf{r}_1-\mathbf{r}_0}{|\mathbf{r}_1-\mathbf{r}_0|} \tag{80}$$

$$\mathbf{v}_1=(A+B)\frac{\mathbf{r}_1-\mathbf{r}_0}{|\mathbf{r}_1-\mathbf{r}_0|}-(B-A)\frac{\mathbf{r}_1}{|\mathbf{r}_1|} \tag{81}$$

where

$$A=\begin{cases} \sqrt{\frac{\mu}{4a}}\cot\frac{\alpha}{2} & E<0 \\ \sqrt{\frac{\mu}{2s}} & E=0 \\ \sqrt{\frac{\mu}{-4a}}\coth\frac{\gamma}{2} & E>0 \end{cases} \tag{82}$$

$$B=\begin{cases} \sqrt{\frac{\mu}{4a}}\cot\frac{\beta}{2} & E<0 \\ \sqrt{\frac{\mu}{2(s-c)}} & E=0 \\ \sqrt{\frac{\mu}{-4a}}\coth\frac{\delta}{2} & E>0 \end{cases} \tag{83}$$

D. Perturbed Two-Body Problem

Consider a perturbing force acting on the system that preserves the Hamiltonian structure of the system. Let this potential be a function of the coordinates only. The Hamiltonian of the perturbed system is

$$H=\frac{1}{2}\mathbf{v}\cdot\mathbf{v}-\frac{\mu}{|\mathbf{r}|}+H^{(1)}(\mathbf{r}) \tag{84}$$

where $H^{(1)}(\mathbf{r})$ is the difference between the perturbed and unperturbed Hamiltonians. Since the Hamiltonian of the system is quadratic in the velocities, the total principal function will be simplified as $\partial^\alpha H/\partial v_i\partial v_j\partial v_k\cdots$, for $\alpha=3,4,\ldots,\infty$ will vanish. Therefore,

$$W^{(1)}=-\int_{t_0}^{t_1}\{H^{(1)}[\mathbf{r}(t)]\,\mathrm{d}t\}$$

$$W^{(\alpha)}=-\frac{1}{2}\int_{t_0}^{t_1}\left[\sum_{\beta=1}^{\alpha-1}\mathbf{v}^{(\beta)}(t)\cdot\mathbf{v}^{(\alpha-\beta)}(t)\,\mathrm{d}t\right] \qquad \alpha=2,\ldots,n \tag{85}$$

where

$$\mathbf{v}^{(1)}(t)=\int_t^{t_1}\frac{\partial H^{(1)}}{\partial q_j(\tau)}\frac{\partial_b q_j(\tau)}{\partial_b q_i(t)}\,\mathrm{d}\tau \tag{86}$$

Recall that the ith component of the initial velocity $\mathbf{v}_0=-(\partial W/\partial\mathbf{r}_0)$. Therefore, the required initial velocity that can be computed in order to solve the 2BVP for the transfer time t_1-t_0 is

$$\mathbf{v}_0^{(1)}=\int_{t_0}^{t_1}\frac{\partial H^{(1)}}{\partial r_j}\frac{\partial_b r_j}{\partial_b r_{1i}}\,\mathrm{d}t$$

$$\mathbf{v}_0^{(\alpha)}=\frac{1}{2}\int_{t_0}^{t_1}\left[\sum_{\beta=1}^{\alpha-1}\left(\frac{\partial v_i^{(\beta)}}{\partial r_{0j}}v_j^{(\alpha-\beta)}+\frac{\partial v_i^{(\alpha-\beta)}}{\partial r_{1j}}v_j^{(\beta)}\right)\right]\mathrm{d}t \tag{87}$$

Figures 2 and 3 illustrate how the perturbation theory derived in the previous section is related to the two-body problem. Consider the Keplerian orbit that connects two points A and B in a given finite time $T=t_b-t_a$. However, in the presence of a perturbing force, these two points will be connected by a different orbit. The figures show the higher-order solutions of the theory converging to the velocity that results in the true solution to the problem $\mathbf{v}^{(\infty)}$.

E. Third-Body Tidal Perturbing Effects

Consider the three-body problem, where a particle orbits two massive bodies in orbit around the center of mass of the system. The most common form of the equations of motion are expressed in the synodic frame, which is a rotating frame where the origin is in the barycenter of the system. Another interesting frame of reference is the inertially fixed frame in one of the massive bodies. This frame offers the possibility to study the motion of the particle with respect to one of the bodies while the other body has a smaller effect on the particle. The equations of motion of the particle with respect to the primary in the inertial frame are

$$\ddot{\mathbf{r}}=-\mu_1\frac{\mathbf{r}}{|\mathbf{r}|^3}+\mu_2\left[\frac{\mathbf{r}_2-\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}-\mathbf{r}_1|^3}-\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}\right] \tag{88}$$

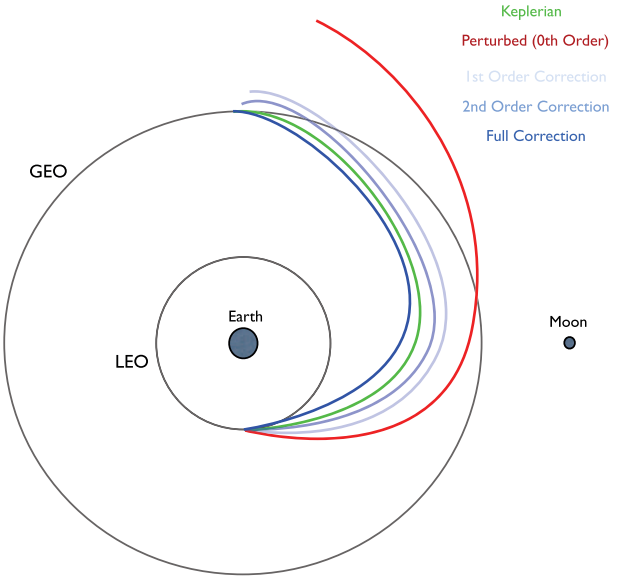

Fig. 2 Illustration of perturbed 2PBVP.

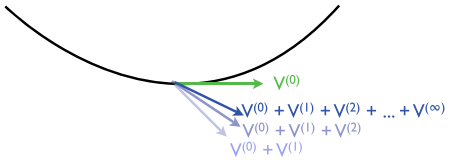

Fig. 3 Detailed illustration of perturbed 2PBVP.

where $\mathbf{r}$ is the position vector of the particle relative to the first body, $\mathbf{r}_1$ is the position vector of the first body with respect to the barycenter, and $\mathbf{r}_2$ is the position vector of the second body with respect to the barycenter.

The three-body system centered at an inertial frame at the center of the primary can be viewed as a perturbed two-body problem, where the perturbation arises due to the presence of the secondary mass, as illustrated by Fig. 4.

To solve the perturbed two-point boundary problem, the perturbing potential can be obtained from Eq. (88) as

$$H^{(1)} = \mu_2\left[\frac{1}{|\mathbf{r}_2 - \mathbf{r} - \mathbf{r}_1|} - \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}\cdot\mathbf{r}\right] \tag{89}$$

Therefore, the acceleration on the particle due to the perturbation of the third body is

$$\ddot{\mathbf{a}} = \mu_2\frac{\mathbf{r}_2 - \mathbf{r} - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r} - \mathbf{r}_1|^3} - \mu_2\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \tag{90}$$

The first term of Eq. (90) is the acceleration of the particle from the third body, and the second term is the acceleration between the two massive bodies.

F. Central Body Oblateness Effects

Consider a particle around a sphere with an oblateness parameter C_{20}. The perturbing force potential U_{C20} that arises due to the oblateness effect is

$$H^{(1)} = U_{C20} = \frac{\mu_1}{2}\frac{C_{20}R_B}{|\mathbf{r}|^5}[\mathbf{r}\cdot\mathbf{r} - 3(\delta_{3i}r_i)^2] \tag{91}$$

where R_B is the mean radius of the central body, and δ_{ki} is the Kronecker delta function

$$\delta_{ki} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases} \tag{92}$$

Since the perturbation from the oblateness effect of the central body can be expressed as a force potential, the Hamiltonian of the perturbed system can be written as

$$H = \frac{\mu_1}{2}\mathbf{v}\cdot\mathbf{v} - \frac{\mu_1}{|\mathbf{r}|} - \frac{\mu_1}{2}\frac{C_{20}R_B}{|\mathbf{r}|^5}[\mathbf{r}\cdot\mathbf{r} - 3(\delta_{3i}r_i)^2] \tag{93}$$

VI. Numerical Solutions to Perturbed Boundary Value Problems

Solutions to the Hohmann transfer and Lambert's problem are special cases of 2BVPs, where the initial and final positions and times are given and the required velocities to connect the two points are to be determined.

Consider a spacecraft orbit transfer between a circular low Earth orbit (LEO) and a circular geostationary orbit (GEO). The fuel optimal two-impulse transfer for the two-body problem can be solved by obtaining the results from the Hohmann transfer equations.

However, these results will not take into account perturbing forces that are actually present and affect the transfer orbit. Two important perturbing forces for this particular case are higher-order gravitational forces from Earth's gravity field and tidal effects from the proximity of the moon.

With perturbing forces present, the previously described perturbation theory can be used to obtain the required change in initial velocity in order to hit the target point, and therefore have a corrected Hohmann transfer. However, this new solution is not necessarily the optimal solution to the perturbed problem. Therefore, what is the optimal two-impulse transfer to go from a LEO to a GEO?

Since perturbations are relatively small, it makes physical sense that the optimal solution for the perturbed problem will be in the vicinity of the nominal solution. Figure 5 illustrates the vicinity of the Hohmann transfer, and the 2BVP can still be solved for a given deviation angle $\Delta\Theta$ and deviation transfer time ΔT_H.

A. Example: Perturbed Low-Earth-Orbit to Geostationary-Orbit Transfer

Consider again a spacecraft in a circular LEO orbit where the desired maneuver is a two-impulse transfer to a circular GEO orbit: one at the beginning of the transfer and one at the end.

The distances are normalized by the Earth–moon semimajor axis $D = 3.844 \times 10^5$ km. Therefore, the initial and final semimajor axes of the circular orbits will be $a_0 = 0.01743$ and $a_1 = 0.11$, respectively. The normalized radius of the Earth is therefore $R_E = 0.01665$.

The masses are normalized by the total mass of the Earth–moon system. Therefore, the gravitational parameter of the Earth and moon will be $\mu_1 = 0.98785$ and $\mu_2 = 0.01215$, respectively. It is assumed that the mass of the spacecraft is negligible.

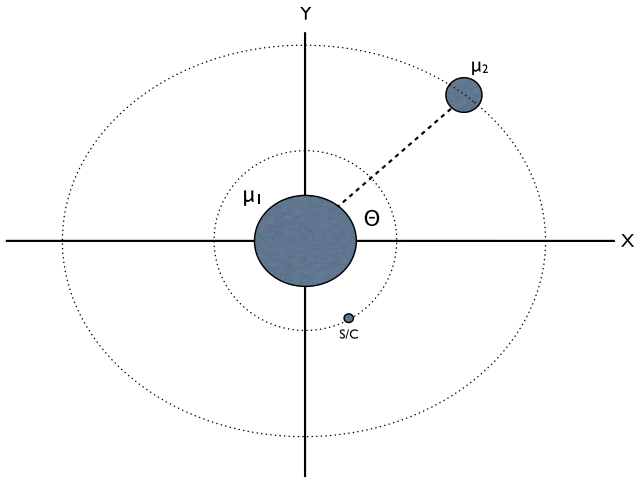

Fig. 4 Illustration of perturbed two-body problem.

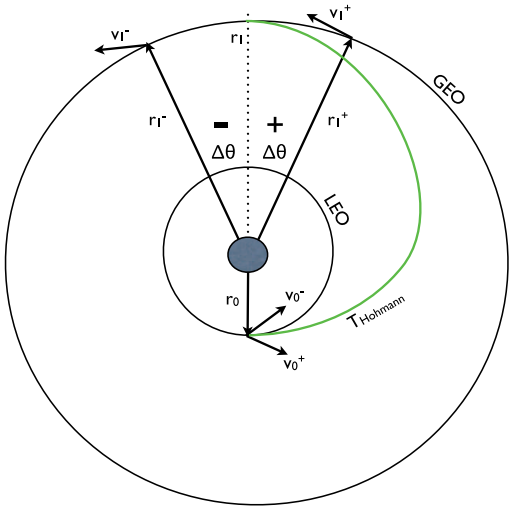

Fig. 5 Illustration of the vicinity of the Hohmann transfer.

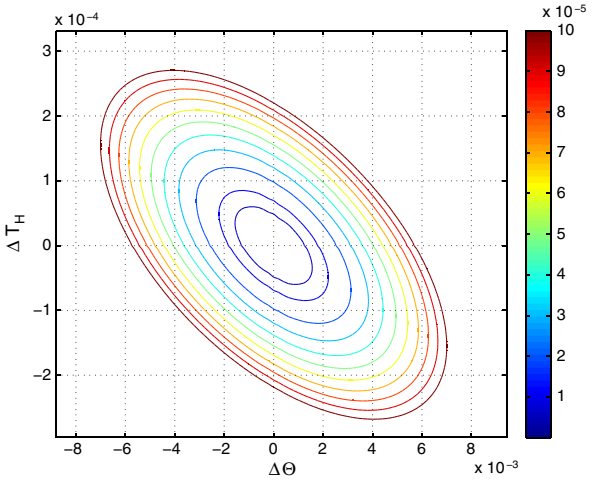

Fig. 7 Detailed ΔV contour map around the Hohmann transfer.

The time parameter is normalized by the moon's orbit period around the Earth, $T_M = 2.3606 \times 10^6$ s. Therefore, from Eq. (78), the Hohmann transfer time from LEO to GEO is $T_H = 0.0508$. The total impulse for this transfer can be obtained from Eq. (76):

$$\Delta V_H = \Delta V_0 + \Delta V_1 = 2.3634 + 1.4294 = 3.7928$$

Figures 6 and 7 illustrate a contour plot of the nominal 2BVP for the two-body problem. The contour lines represent the total ΔV cost of the LEO to GEO transfer for different coordinates $(\Delta\theta, \Delta T_H)$. These coordinates represent the deviation in the transfer angle and transfer time from the Hohmann transfer. Note that, despite this deviation, the transfer is still a LEO to GEO transfer; the goal is now finding solutions to the Lambert's problem instead of solving the Hohmann transfer. This is illustrated in Fig. 5

For the Keplerian transfer, where no perturbing forces are present, it is expected that the least costly two-impulse transfer will be the Hohmann transfer. Figure 6 shows that this is the case on a larger scale where nonlinearities are present.

Figure 7 shows a more detailed illustration around the Hohmann transfer. The region around the nominal solution shows linear variations around the origin due to the very close proximity to the origin. Dynamics in this case are in the linear region, as the range of values for $\Delta\theta$ and ΔT_H are very small. Recall that $T_H = 0.0508$ for the LEO to GEO Hohmann transfer. The contour lines indicate the deviation from the total Hohmann transfer ΔV cost. As expected, the minimum occurs at the Hohmann transfer point $(\Delta\theta, \Delta T_H) = (0, 0)$, and it increases as we deviate from the nominal solution.

The moon's phase angle with respect to the inertial frame centered at the Earth will influence the results obtained from the perturbation theory. The tidal effect of the moon's gravity on the spacecraft will differ as a function of the moon's position with respect to the spacecraft. The initial phase angle of the moon is chosen to be $\Phi = \pi/2$, since this phase angle has the most significant influence of all quarter-phase angles $(0, \pi/2, \pi, 3\pi/2)$. Earth's higher-order gravitational field will also cause the actual results to differ from the nominal solution. Earth's oblateness parameter is $C_{20} = -0.001$.

Figure 8 shows the total ΔV contour of the corrected transfer in the presence of perturbing forces. The theory is used to find the necessary change in ΔV to solve the perturbed 2BVP. As can be seen, the contour has now shifted to another region of $(\Delta\theta, \Delta T_H)$. The total ΔV of the transfer is also less than the Hohmann transfer ΔV. This implies that, depending on the perturbing forces and their effects, a more cost-efficient transfer can be found using the theory and the subsequent approach to solving 2BVPs.

B. Physical Explanation of Results

Figure 7 shows smooth and consistent yet tilted ellipsoids around the Hohmann transfer solution $(\Delta\theta, \Delta T_H = 0, 0)$. Although, at first, this tilt might not appear intuitive, it makes perfect sense from a dynamical standpoint.

The figure shows that if the transfer angle is less than 180 deg $(\Delta\theta < 0)$, the optimal transfer time must be shorter than the Hohmann transfer time, because it will require less change in energy to get there. Since the target final position is closer to the initial point, the time to get there should be less than the Hohmann transfer time.

Conversely, if the desired transfer angle is greater than 180 deg $(\Delta\theta > 0)$, a longer transfer will require less change in energy, making it more fuel efficient for two-impulse transfers. The further the target

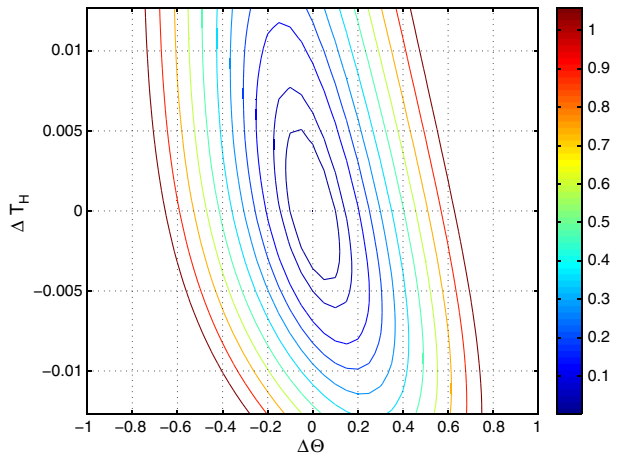

Fig. 6 ΔV contour map around the Hohmann transfer.

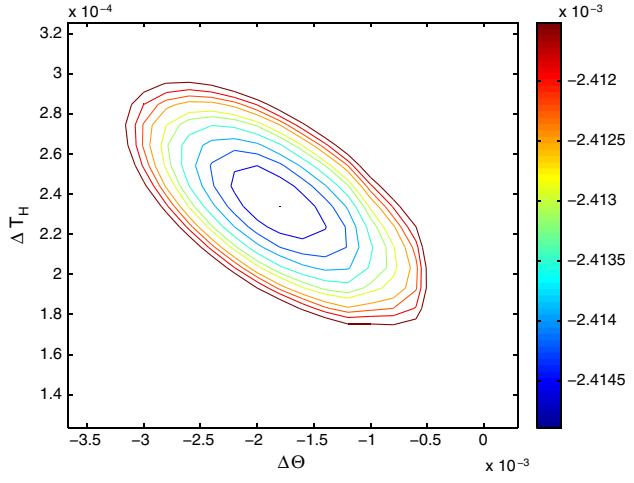

Fig. 8 Detailed ΔV contour map of perturbed 2BVP solution.

point is from the Hohmann transfer, the greater the transfer time to get there.

Figure 8 shows the total ΔV contour of the perturbed two-body 2BVP. The new fuel-optimal target point is now located at

$$(\Delta\theta, \Delta T_H) = (-0.00175, -0.00023)$$

which implies that the solution has shifted to a target point further away from the Hohmann transfer final position, and the transfer time is greater than T_H.

Note that these results are only an example, as the perturbation effects depend on the phase angle of the moon. Different phase angles will offer different results. However, for this particular case ($\Phi = \pi/2$), the shift in transfer angle and time also offer a more fuel optimal transfer than the Hohmann transfer. The contour lines indicate the relative total cost of the transfer. The transfer cost for this example is $\Delta V = -0.002415$, which is about 0.06% lower cost than the nominal Hohmann transfer cost. The total cost of the perturbed Hohmann transfer is $\Delta V = 3.7905$; therefore, the perturbation method offers a savings.

C. Example: Numerical Errors in Low-Earth-Orbit to Geostationary-Orbit Transfer

Consider again a spacecraft going from LEO to GEO on a Hohmann transfer. The oblateness parameter of the Earth is increased by a factor of 100 to show how the theory performs in a highly perturbed environment. For the zeroth-order solution, the final position error is 2.4516*e*–4 km, and the relative error of the initial velocity is 0.0720.

Figure 9 shows the relative error in the calculation of the initial velocity for varying degrees of time-step-size calculations for first-order and second-order corrections using the perturbation theory developed in this paper. It is interesting to note that, in order to obtain accurate results, small time steps must be taken in the quadrature scheme. A small time-step size is an intuitive approach because, as the order of calculation in increased, the correction term will be smaller for each order, resulting in more sensitive numerical accuracies.

Figure 10 shows the error in the final spacecraft target as the integration step sizes vary for first- and second-order calculations for this highly perturbed example. The errors for both first and second orders level off for $\Delta t_2 = 10^{-4}$; however, Δt_1 is at least an order of magnitude smaller for these cases.

These results imply that the first-order results need to be calculated very precisely in order to have accurate second-order results. Second-order calculations depend directly on the first order, so the larger the error in the first order, the worse the result. This can be seen for the case of $\Delta t_1 = 10^{-4}$, where the initial velocity calculations actually diverge with the second order.

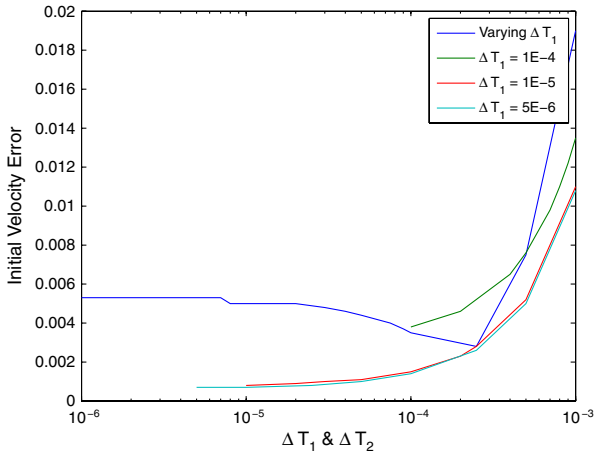

Fig. 9 Semilog plot of initial velocity error.

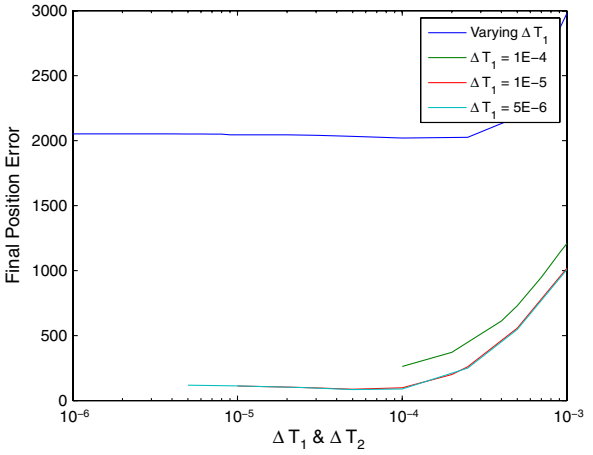

Fig. 10 Final position error semilog plot.

VII. Conclusions

A perturbation theory that allows one to analytically solve 2BVPs for a Hamiltonian dynamical system has been presented. This theory is based on HPF, which allows one to solve for the motion of the system in phase space.

There are many potential applications for the theory that include orbit transfers, configuration of spacecraft formations, and optimal control problems. Moreover, the theory might find applications in other areas of engineering and science.

It has been shown how the theory can be applied to the perturbed two-body problem. Second-order approximations yield accurate results for strong perturbations over short transfer times. However, higher-order approximations must be implemented to accurately approximate longer transfer orbits in highly perturbed environments.

A numerical error analysis of the theory to better understand the precision needed to solve for the quadratures is also introduced. This analysis shows the importance of obtaining the first-order solution with as much precision as possible.

The theory relates to von Zeipel's method at a basic level. Both methods expand the system around the nominal solution through a small parameter ϵ. However, the method does not need a reduction in the Hamiltonian in order to analyze the system.

Appendix A: Hamilton's Perturbation Theory

Hamilton developed a first-order perturbation theory for a general Hamiltonian dynamical system that yields an approximate solution to the IVP [9]. This technique assumes that the unperturbed system is Hamiltonian and that its solution is known. Consider a Hamiltonian dynamical system of degree n and separate the principal function

$$W = W^{(0)}(\mathbf{q}_0^{(0)}, \mathbf{q}_1^{(0)}, t_0, t_1) + W^{(1)}(\mathbf{q}_0, \mathbf{q}_1, t_0, t_1) \qquad \text{(A1)}$$

where $W^{(0)}$ is the principal function for the unperturbed solution and $W^{(1)}$ is the correction that allows W to be the principal function of the actual system. The Hamiltonian function can also be separated into the Hamiltonian of the unperturbed solution $H^{(0)}$ and the correction $H^{(1)}$ as

$$H(\mathbf{q}, \mathbf{p}) = H^{(0)}(\mathbf{q}^{(0)}, \mathbf{p}^{(0)}) + H^{(1)}(\mathbf{q}, \mathbf{p}) \qquad \text{(A2)}$$

where $(\mathbf{q}^{(0)}, \mathbf{p}^{(0)})$ represents the state of the unperturbed solution. From Eqs. (8), (9), and (11), the principal function will satisfy

$$\mathbf{p}_0 = -\frac{\partial W^{(0)}}{\partial \mathbf{q}_0} - \frac{\partial W^{(1)}}{\mathbf{q}_0} \qquad \mathbf{p}_1 = \frac{\partial W^{(0)}}{\partial \mathbf{q}_1} + \frac{\partial W^{(1)}}{\mathbf{q}_1} \qquad \text{(A3)}$$

and

$$-\frac{\partial W^{(0)}}{\partial t_0}-\frac{\partial W^{(1)}}{\partial t_0}+H^{(0)}(\mathbf{q}_0^{(0)},\mathbf{p}_0^{(0)},t_0)+H^{(1)}(\mathbf{q}_0,\mathbf{p}_0,t_0)=0$$

$$\frac{\partial W^{(0)}}{\partial t_1}+\frac{\partial W^{(1)}}{\partial t_1}+H^{(0)}(\mathbf{q}_1^{(0)},\mathbf{p}_1^{(0)},t_1)+H^{(1)}(\mathbf{q}_1,\mathbf{p}_1,t_1)=0 \quad \text{(A4)}$$

Since $W^{(0)}$ is the principal function for the unperturbed solution, the following relationships hold:

$$\mathbf{p}_0^{(0)}=-\frac{\partial W^{(0)}}{\partial \mathbf{q}_0} \qquad \mathbf{p}_1^{(0)}=\frac{\partial W^{(0)}}{\partial \mathbf{q}_1} \quad \text{(A5)}$$

and

$$-\frac{\partial W^{(0)}}{\partial t_0}+H^{(0)}\left(\mathbf{q}_0^{(0)},-\frac{\partial W^{(0)}}{\partial \mathbf{q}_0},t_0\right)=0$$

$$\frac{\partial W^{(0)}}{\partial t_1}+H^{(0)}\left(\mathbf{q}_1^{(0)},\frac{\partial W^{(0)}}{\partial \mathbf{q}_1},t_1\right)=0 \quad \text{(A6)}$$

Therefore,

$$\Delta\mathbf{p}_0=-\frac{\partial W^{(1)}}{\partial \mathbf{q}_0} \qquad \Delta\mathbf{p}_1=\frac{\partial W^{(1)}}{\partial \mathbf{q}_1} \quad \text{(A7)}$$

and

$$\frac{\partial W^{(1)}}{\partial t_0}-H^{(1)}\left(\mathbf{q}_0,-\frac{\partial W}{\partial \mathbf{q}_0},t_0\right)=0$$

$$\frac{\partial W^{(1)}}{\partial t_1}+H^{(1)}\left(\mathbf{q}_1,\frac{\partial W}{\partial \mathbf{q}_1},t_1\right)=0 \quad \text{(A8)}$$

where Δ indicates the correction to the unperturbed system. From Hamiltonian dynamics, the equations of motion of the system satisfy

$$\dot{\mathbf{q}}=\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial H^{(0)}}{\partial \mathbf{p}}+\frac{\partial H^{(1)}}{\partial \mathbf{p}} \quad \text{(A9)}$$

$$\dot{\mathbf{p}}=-\frac{\partial H}{\partial \mathbf{q}}=-\frac{\partial H^{(0)}}{\partial \mathbf{q}}-\frac{\partial H^{(1)}}{\partial \mathbf{q}} \quad \text{(A10)}$$

Hamilton argues that if the Hamiltonian associated with the perturbation $H^{(1)}$ and its coefficients are sufficiently small, then the equations of motion can be approximated as

$$\dot{\mathbf{q}}\approx\frac{\partial H^{(0)}}{\partial \mathbf{p}} \quad \text{(A11)}$$

$$\dot{\mathbf{p}}\approx-\frac{\partial H^{(0)}}{\partial \mathbf{q}} \quad \text{(A12)}$$

The total time derivative of the principal function can be expressed as [11]

$$\frac{\mathrm{d}W}{\mathrm{d}t_k}=\frac{\partial W}{\partial t_k}+\frac{\partial W}{\partial q_k}\frac{\partial q_k}{\partial t_k} \quad \text{(A13)}$$

With the aid of the preceding result, and after several algebraic manipulations, the correction to the unperturbed principal function can be expressed as

$$W^{(1)}=-\int_0^t H^{(1)}\,\mathrm{d}t \quad \text{(A14)}$$

Therefore, in order to calculate the configuration of the actual system at any time, t can be approximated by the following expressions [9]:

$$\Delta p_i=-\int_0^t \frac{\partial H^{(1)}}{\partial q_{0i}}\,\mathrm{d}t \quad \text{(A15)}$$

$$\Delta\mathbf{q}=-\int_0^t \frac{\partial H^{(1)}}{\partial \mathbf{p}_0}\,\mathrm{d}t \quad \text{(A16)}$$

where the position and velocity vector at time t will be

$$\mathbf{q}=\mathbf{q}^{(0)}+\Delta\mathbf{q} \qquad \mathbf{p}=\mathbf{p}^{(0)}+\Delta\mathbf{p} \quad \text{(A17)}$$

The preceding result provides the approximate expressions for the states at time t that are subject to a first-order perturbation. Note that this perturbation technique assumes that the perturbation is small and that the perturbed equations of motion can be approximated by the unperturbed ones.

If an analytical solution to the first-order perturbation is found, one can redefine the perturbed system as unperturbed and treat a second-order perturbation as a first-order perturbation to this new system. Higher-order perturbed systems can be solved in a similar manner.

Appendix B: Hamilton's Principal Function for Two-Body Problem and Related Results

Theorem 1.1: For a specific energy level E, the HPF for the two-body problem is

$$W(\mathbf{r}_0,\mathbf{r}_1,t_0,t_1)=\begin{cases}\sqrt{\mu a}[(\alpha+\sin\alpha)-(\beta+\sin\beta)]+\frac{\mu}{2a}(t_1-t_0) & E<0\\ \sqrt{8\mu}\left[\sqrt{\frac{s}{2}}-\sqrt{\frac{s-c}{2}}\right] & E=0\\ \sqrt{-\mu a}[(\delta+\sinh\delta)-(\gamma+\sinh\gamma)]+\frac{\mu}{-2a}(t_1-t_0) & E>0\end{cases} \quad \text{(B1)}$$

where

$$\sin\frac{\alpha}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|+|\mathbf{r}_0+\mathbf{r}_1|}{4a}}$$

$$\sin\frac{\beta}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|-|\mathbf{r}_0+\mathbf{r}_1|}{4a}}$$

$$\sinh\frac{\gamma}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|+|\mathbf{r}_0+\mathbf{r}_1|}{-4a}}$$

$$\sinh\frac{\delta}{2}=\sqrt{\frac{|\mathbf{r}_0|+|\mathbf{r}_1|-|\mathbf{r}_0+\mathbf{r}_1|}{-4a}}$$

$$s=\frac{|\mathbf{r}_0|+|\mathbf{r}_1|+|\mathbf{r}_0+\mathbf{r}_1|}{2}$$

$$c=|\mathbf{r}_0+\mathbf{r}_1|$$

and the semimajor axis a is related to the total energy/Hamiltonian of the system by

$$a=-\frac{\mu}{2E}$$

Proof: (See [15] for a detailed derivation.) First, it is necessary verify that W satisfies the two partial differential equations. Differentiating W with respect to t_0 and t_1, we obtain

$$\frac{\partial W}{\partial t_0}=\begin{cases}-\frac{\mu}{2a} & E<0\\ 0 & E=0\\ \frac{\mu}{2a} & E>0\end{cases} \quad \text{(B2)}$$

$$\frac{\partial W}{\partial t_1}=\begin{cases}\frac{\mu}{2a} & E<0\\ 0 & E=0\\ -\frac{\mu}{2a} & E>0\end{cases} \quad \text{(B3)}$$

Therefore, Eqs. (B2) and (B3) satisfy the two necessary partial differential equations. Next, is shown that W satisfies the two boundary conditions

$$\frac{\partial W}{\partial \mathbf{r}_0} = (A-B)\frac{\mathbf{r}_0}{|\mathbf{r}_0|} - (A+B)\frac{\mathbf{r}_1-\mathbf{r}_0}{|r_1-r_0|} \tag{B4}$$

$$\frac{\partial W}{\partial \mathbf{r}_1} = (A+B)\frac{\mathbf{r}_1-\mathbf{r}_0}{|\mathbf{r}_1-\mathbf{r}_0|} - (B-A)\frac{\mathbf{r}_1}{|\mathbf{r}_1|} \tag{B5}$$

where

$$A = \begin{cases} \sqrt{\frac{\mu}{4a}}\cot\frac{\alpha}{2} & E<0 \\ \sqrt{\frac{\mu}{2s}} & E=0 \\ \sqrt{\frac{\mu}{-4a}}\coth\frac{\gamma}{2} & E>0 \end{cases} \tag{B6}$$

$$B = \begin{cases} \sqrt{\frac{\mu}{4a}}\cot\frac{\beta}{2} & E<0 \\ \sqrt{\frac{\mu}{2(s-c)}} & E=0 \\ \sqrt{\frac{\mu}{-4a}}\coth\frac{\delta}{2} & E>0 \end{cases} \tag{B7}$$

To verify that this W indeed satisfies the boundary conditions, compare Eqs. (B4) and (B5) to the solution to the 2BVP. A detailed procedure using geometry can obtain [2,16]

$$\mathbf{v}_0 = (B-A)\frac{\mathbf{r}_0}{|\mathbf{r}_0|} + (A+B)\frac{\mathbf{r}_1-\mathbf{r}_0}{|\mathbf{r}_1-\mathbf{r}_0|} \tag{B8}$$

$$\mathbf{v}_1 = (A+B)\frac{\mathbf{r}_1-\mathbf{r}_0}{|\mathbf{r}_1-\mathbf{r}_0|} - (B-A)\frac{\mathbf{r}_1}{|\mathbf{r}_1|} \tag{B9}$$

It becomes apparent that Eq. (B4) is the negative of Eq. (B5), and Eq. (B8) is exactly equal to Eq. (B9). From Hamilton's theorems, Eq. (B4) has to be the negative of the initial velocity and Eq. (B5) has to be equal to the final velocity. Therefore, the proposed W function satisfies the required conditions to be the HPF of the two-body problem.

Appendix C: Lambert's Theorem and Hamilton's Principal Function

Recall Lambert's equation [17]

$$\sqrt{\mu}(t_1-t_0) = \begin{cases} a^{3/2}[(\alpha-\sin\alpha)-(\beta-\sin\beta)] & E<0 \\ \sqrt{\frac{2}{9}}[(s)^{3/2}-(s-c)^{3/2}] & E=0 \\ (-a)^{3/2}[(\sinh\gamma-\gamma)-(\sinh\delta-\delta)] & E>0 \end{cases} \tag{C1}$$

The system lies on the surface $E=H$ in the extended phase space. Thus, the gradient of HPF with respect to the energy on that surface vanishes [5]. Setting $\partial W/\partial E = 0$ for the elliptic and hyperbolic cases, we find that

$$t_1-t_0 = \begin{cases} \sqrt{\frac{a^3}{\mu}}[(\alpha-\sin\alpha)-(\beta-\sin\beta)] & E<0 \\ \sqrt{\frac{-a^3}{\mu}}[(\sinh\gamma-\gamma)-(\sinh\delta-\delta)] & E>0 \\ \sqrt{\frac{2}{9\mu}}[(s)^{3/2}-(s-c)^{3/2}] & E=0 \end{cases} \tag{C2}$$

Thus, Lambert's equation can be obtained from the principal function. While Lambert and Lagrange based their findings on spatial geometry, this shows that it is possible to derive the result from a variational point of view.

Appendix D: Additional Properties of Hamilton's Principal Function

HPF transforms the state of the system $\mathbf{q}_0, \mathbf{p}_0$ at a time t_0 into some later state $\mathbf{q}_1, \mathbf{p}_1$ at time t_1. In addition, for a conservative system, our variables $\mathbf{q}$ and $\mathbf{p}$ are restricted to the energy surface $E=H$ and cannot leave that surface during the transformation. That is, if $\mathbf{q}_0, \mathbf{q}_1$, and t_1-t_0 are unchanged, then the energy, a direct function of these variables, is unchanged. This implies that the gradient of W with respect to the energy is zero [5]:

$$\frac{\partial W}{\partial E} = 0 \tag{D1}$$

Since HPF involves moving endpoints and an energy surface, it generates a transformation that maps a point on the energy surface $E=H$ of the extended phase space to another point on the same surface. Since there exists a constraint on how HPF changes as a function of q_{0i} and q_{1i}, the following condition holds [5]:

$$\left|\frac{\partial^2 W}{\partial \mathbf{q}_0 \partial \mathbf{q}_1}\right| = 0 \tag{D2}$$

Theorem 2.1: The velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ are the left and right eigenvectors corresponding to the null space of $\partial^2 W/\partial \mathbf{r}_0 \partial \mathbf{r}_1$.

Proof:

$$\frac{\partial^2 W}{\partial r_{0i}\partial r_{1j}} v_{1j} = \frac{\partial^2 W}{\partial r_{0i}\partial r_{1j}}\frac{\partial W}{\partial r_{1j}} \tag{D3}$$

The preceding equation can also be expressed as

$$\frac{\partial^2 W}{\partial r_{0i}\partial r_{1j}} v_{1j} = \frac{\partial}{\partial r_{0j}}\left[\frac{1}{2}\frac{\partial W}{\partial r_{0j}}\frac{\partial W}{\partial r_{1i}}\right] = \frac{\partial}{\partial r_{0j}}\left[\frac{1}{2}v_{0j}v_{1i}\right] \tag{D4}$$

Recall the energy equation for the two-body problem

$$E = \frac{1}{2}\mathbf{v} - \frac{\mu}{|\mathbf{r}|} \tag{D5}$$

Therefore,

$$\frac{\partial}{\partial r_{0j}}\left[\frac{1}{2}v_{1j}v_{1i}\right] = \frac{\partial}{\partial r_{0i}}\left[E + \frac{\mu}{|r_{1i}|}\right] = 0 \tag{D6}$$

as $\mathbf{r}_0$ and $\mathbf{r}_1$ are independent and the energy is fixed. Thus,

$$\frac{\partial^2 W}{\partial r_{0i}\partial r_{1j}} v_{1j} = 0 \tag{D7}$$

Similarly,

$$v_{0j}\frac{\partial^2 W}{\partial r_{0j}\partial r_{1i}} = \frac{\partial}{\partial r_{1j}}\left[\frac{1}{2}\frac{\partial W}{\partial r_{0j}}\frac{\partial W}{\partial r_{0i}}\right] = \frac{\partial}{\partial r_{1j}}\left[\frac{1}{2}v_{0j}v_{0i}\right] \tag{D8}$$

Substituting the relationship given by Eq. (D5) yields

$$\frac{\partial}{\partial r_{1j}}\left[\frac{1}{2}v_{0j}v_{0i}\right] = \frac{\partial}{\partial r_{1i}}\left[E + \frac{\mu}{|r_{0i}|}\right] = 0 \tag{D9}$$

and

$$v_{0j}\frac{\partial^2 W}{\partial r_{0j}\partial r_{1i}} = 0 \tag{D10}$$

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